Generalizando la Lógica de simplificaciones

P. Cordero, M. Enciso, **A. Mora**

Department of Applied Mathematics - University of Málaga, Spain

V. Vychodil

Dept. Computer Science - Palacky University Olomouc, Czechia

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Worcloud of contents

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Table of Contents

Armstrong's axiomatic system

The set of all valid implications in a context satisfies the well-known Armstrong's axioms:

$$[\texttt{Ref}]\ \frac{A \supseteq B}{A \mapsto B}, \qquad [\texttt{Augm}]\ \frac{A \mapsto B}{AC \mapsto BC}, \qquad [\texttt{Trans}]\ \frac{A \mapsto B,\ B \mapsto C}{A \mapsto C}$$

Armstrong, W.W.: Dependency structures of data base relationships. In: Rosenfeld, J.L., Freeman, H. (eds.) Information Processing 74: Proceedings of IFIP Congress. pp. 580–583. North Holland, Amsterdam (1974)

Simplification Logic

Axioma: $\vdash X \mapsto Y$, si $Y \subseteq X$ (Ax)

Fragmentation: $X \mapsto Y \vdash X \mapsto Y'$ if $Y' \subseteq Y$, (Frag)

Composition: $X \mapsto Y,\ U \mapsto V \vdash XU \mapsto YV$, (Comp)

Simplification: $X \mapsto Y,\ U \mapsto V \vdash (U - Y) \mapsto (V - Y)$, if $X \subseteq U, X \cap Y = \varnothing$. (Simp)

- Cordero, P., Enciso, M., Mora, A., Pérez de Guzman, I.: SLFD Logic: Elimination of Data Redundancy in Knowledge Representation 2527, 141–150 (2002)
- Cordero, P., Enciso, M., Mora, A., Ojeda-Aciego, M.: Computing minimal generators from implications: a logic-guided approach. In: Proc. of Concept Lattices and Applications, CLA 2012, pp. 187–198 (2012)
- E. Rodríguez Lorenzo, P. Cordero, M. Enciso, Á. Mora : Canonical dichotomous direct bases. Information Sciences 376: 39-53 (2017)

What happen with these formulas?

$$A \Rightarrow B,$$

where A and B are graded sets of attributes, such as

$$\left\{{}^{0.2}/y_1, y_2\right\} \Rightarrow \left\{{}^{0.8}/y_3\right\}.$$

Goal

Define general methods to characterize the semantic of different if-then rules: using general lattice-theoretic notions and proposing more general logics to deal with these rules.

First approach

complete residuated lattice

$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, where

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice,
- $\otimes \colon L \times L \to L$ (a multiplication) is commutative, associative, and 1-neutral,
- $\to \colon L \times L \to L$ (a residuum):

$$a \otimes b \leq c \text{ iff } a \leq b \to c \qquad \text{for all } a, b, c \in L.$$

L-sets (fuzzy sets)

considered as maps $A : U \to L$ such that

- U is a (non-empty) universe set
- L is the set of degrees from $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$
- $A(u) =$ "degree to which u belongs to A"

Semantic of $A \Rightarrow B$

We consider the **complete residuated lattice of L-sets in *U***

$$\mathbf{L}^U = \langle L^U, \cap, \cup, \otimes, \rightarrow, 0_U, 1_U \rangle$$

Importance of subsethood

- $A \subseteq B$ **iff** $A(u) \leq B(u)$ **for all** $u \in U$ **(ordinary-style subsethood)**
- $\mathbb{S}(A, B) = \bigwedge_{u \in U}(A(u) \rightarrow B(u))$ **(graded-style subsethood) easy to see:** $\mathbb{S}(A, B) = 1$ **iff** $A \subseteq B$

$||A \Rightarrow B||_{I_x} = S(A, I_x) \rightarrow S(B, I_x)$ **(graded-style subsethood) and in a context** $\langle X, Y, I \rangle$ **with graded attributes:**

$$||A \Rightarrow B||_{\langle X,Y,I \rangle} = \bigwedge_{x \in X} ||A \Rightarrow B||_{I_x}$$

Hedges

truth-stressing hedges

(idempotent) truth-stressing hedge on **L** is map $^*: L \to L$ satisfying:

$$1^* = 1, \quad a^* \leq a, \quad (a \to b)^* \leq a^* \to b^*, \quad a^{**} = a^*$$

two borderline hedges: identity and globalization.

1. **identity: $a^* = a$ for all $a \in L$;**
2. **globalization: $a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$ ... interpreted as "fully true"**

Graded Attribute Implications using Hedges

Y . . . finite set of attributes

Definition

Formula $A \Rightarrow B$ where $A, B \in L^Y$ is graded (fuzzy) attribute implication.

various way to define semantics based on

- graded object attribute incidence data (attribute implications)
- (ranked) data tables over domains with similarities (functional deps.)

Definition

Let $A, B, M \in L^Y$. The degree $||A \Rightarrow B||_M$ to which $A \Rightarrow B$ is true in M:

$$||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M).$$

Fuzzy Attributes Simplification Logic - FASL

The dependencies-implications-"if-then-rules" are expressed by formulas of the form

$$A \Rightarrow B, \tag{1}$$

where A and B are graded sets of attributes, such as

$$\{{}^{0.2}/y_1, y_2\} \Rightarrow \{{}^{0.8}/y_3\}. \tag{2}$$

FASL

Axioma: $\vdash AB \Rightarrow A$ (Ax)

Simplification: $A \Rightarrow B, C \Rightarrow D \vdash A(C - B) \Rightarrow D,$ (Simp)

Multiplication: $A \Rightarrow B \vdash c^{*} \otimes A \Rightarrow c^{*} \otimes B,$ (Mul)

Belohlavek, R., Cordero, P., Enciso, M., Mora, A., Vychodil, V.: Automated prover for attribute dependencies in data with grades. International Journal of Approximate Reasoning 70, 51–67 (2016)

Motivation

semantics of attribute implications:

What influences the meaning of (fuzzy) attribute implications?

- $\mathbf{L}$ … structure of truth degrees
- * … idempotent truth-stressing hedge

Q: *Are there other (reasonable) and general approaches?*

Armstrong-style inference systems:

What extends the fuzzy logics?

Like-Armstrong extension

- $\vdash AB \Rightarrow B$
- $A \Rightarrow B, BC \Rightarrow D \vdash A \cup C \Rightarrow D$
- $A \Rightarrow B \vdash c^* \otimes A \Rightarrow c^* \otimes B$

Fuzzy Attribute Simplification Logic - FASL

- $\vdash AB \Rightarrow A$
- $A \Rightarrow B, C \Rightarrow D \vdash A(C - B) \Rightarrow D$
- $A \Rightarrow B \vdash c^* \otimes A \Rightarrow c^* \otimes B$

Q: *Are there more general extensions?*

$$\frac{A \Rightarrow B}{\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)}$$ for any permissible $\boldsymbol{f}$

Algebraic framework

Complete coresiduated lattice

$$\mathbf{L} = \langle L, \leq, \oplus, \ominus, 0, 1 \rangle$$

where

- $\langle L, \leq 0, 1 \rangle$ is a complete lattice
- 0 is the least element
- 1 es the greatest element
- $\vee$ and $\wedge$ the suprema and infima respectively.
- $\langle L, \oplus, 0 \rangle$ is a commutative monoid.
- $\langle \oplus, \ominus \rangle$ satisfies the adjointess property:

$$\text{For all } a, b, c \in L, \quad a \leq b \oplus c \text{ if and only if } a \ominus b \leq c.$$

Parameterizing

Parameterizations in complete lattices

Let $\mathbb{L} = \langle L, \leq \rangle$ be a complete lattice. Any set S of isotone Galois connections $\langle \boldsymbol{f}, \boldsymbol{g} \rangle$ in $\mathbb{L}$ such that $\langle \mathbf{1}, \mathbf{1} \rangle \in S$ is called an ***L*-parameterization**.
S is said to be an $\mathbb{L}$-parameterization if it is *closed under compositions*, i.e., if $\mathbf{S} = \langle S, \circ, \langle \mathbf{1}, \mathbf{1} \rangle \rangle$ is a monoid.

Motivated by...

- **universality of 1-truth**: $||A \Rightarrow B||_M = \bigvee \{c \in L;\ ||A \Rightarrow c \otimes B||_M = 1\}$
- $A \Rightarrow B$ **is fully true**: $||A \Rightarrow B||_M = 1$ iff for any $c \in L$: $c^* \otimes A \subseteq M$ implies $c^* \otimes B \subseteq M$

Parameterized Inference System

$$\mathit{Axiom}: \text{Infer } A \cup B \Rightarrow A, \text{ for any } A, B \in L^Y \qquad \texttt{(Ax)}$$

$$\mathit{Pseudotransitivity}: \frac{A \Rightarrow B,\ B \cup C \Rightarrow D}{A \cup C \Rightarrow D} \quad \text{for all } A, B, C, D \in L^Y, \qquad \texttt{(Pseud)}$$

$$\mathit{Extension}: \frac{A \Rightarrow B}{\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)} \quad \text{for some } \langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S. \qquad \texttt{(Ext)}$$

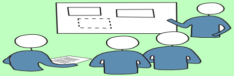
considering

- $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$.
- truth-stressing hedge on **L**.
- a complete residuated lattice of **L**-sets in $\boldsymbol{U}$: $\mathbf{L}^U = \langle L^U, \cap, \cup, \otimes, \rightarrow, 0_U, 1_U \rangle$
- considering an $\boldsymbol{L}$-parameterization and S a set of (lower) adjoints of isotone Galois connections.

Vychodil, V.: Parameterizing the semantics of fuzzy attribute implications by systems of isotone Galois connections. IEEE Trans. on Fuzzy Systems 24, 645–660 (2017)

Generalization

Simplification Paradigm



Armstrong like

Axiom: Infer $A \cup B \Rightarrow A$, for any $A, B \in L^Y$

Pseudotransitivity: $\dfrac{A \Rightarrow B,\ B \cup C \Rightarrow D}{A \cup C \Rightarrow D}$ for all $A, B, C, D \in L^Y$,

Extension: $\dfrac{A \Rightarrow B}{\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)}$ for some $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$.

Vychodil, 2017

Fuzzy

FASL

$\vdash AB \Rightarrow A$

$A \Rightarrow B, C \Rightarrow D \vdash A(C - B) \Rightarrow D$

$A \Rightarrow B \vdash c^* \otimes A \Rightarrow c^* \otimes B$

Belohlavek, Cordero Enciso, Mora Vychodil, 2016

Armstrong like

$\vdash AB \Rightarrow B$

$A \Rightarrow B, BC \Rightarrow D \vdash A \cup C \Rightarrow D$

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Belohlavek, Vychodil

Crisp

Simplification Logic

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Cordero, Enciso Guzmán, Mora

Armstong's Axioms

[Ref] $\dfrac{A \supseteq B}{A \mapsto B}$, [Augm] $\dfrac{A \mapsto B}{AC \mapsto BC}$, [Trans] $\dfrac{A \mapsto B,\ B \mapsto C}{A \mapsto C}$

Armstrong

The language of the logic

The language

Given a non-empty *alphabet* Y, whose elements are named *attributes*, the set of well-formed formulas of the language is:

$$\mathcal{L}_Y = \{A \Rightarrow B \mid A, B \in L^Y\}.$$

The semantic of the logic

Definition

Let Y be a non-empty set and S be an $\mathbb{L}$-parameterization.
An $\mathbb{L}$-fuzzy set $A \in L^Y$ is said to be *S-additive* if $\boldsymbol{f}(B) \subseteq A$ and $\boldsymbol{f}(C) \subseteq A$ imply $\boldsymbol{f}(B \oplus C) \subseteq A$, for all $B, C \in L^Y$ and $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$.

Proposition

Let Y be a non-empty set and S be an $\mathbb{L}$-parameterization. An $\mathbb{L}$-fuzzy set $A \in L^Y$ is S-additive if and only if $\boldsymbol{g}(A) \oplus \boldsymbol{g}(A) = \boldsymbol{g}(A)$.

The semantic of the logic

Given S an $\mathbb{L}$-parameterization, the models of the logic are defined in terms of S-additive $\mathbb{L}$-sets as follows:

Model

Let $A \Rightarrow B \in \mathcal{L}_Y$. An S-additive set $M \in L^Y$ is said to be a *model* for $A \Rightarrow B$, denoted by $M \models A \Rightarrow B$, if

$$\boldsymbol{f}(A) \subseteq M \text{ implies } \boldsymbol{f}(B) \subseteq M, \text{ for all } \langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$$

- The set of models for $A \Rightarrow B$ is denoted by $\mathcal{M}od(A \Rightarrow B)$.
- $\mathcal{M}od(\Sigma) = \bigcap_{A \Rightarrow B \in \Sigma} \mathcal{M}od(A \Rightarrow B)$.

Inference System

SL for Graded Attribute Implications

For all $A, B, C, D \in L^Y$ and $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$, the inference system consists of following axiom scheme:

Reflexivity: Infer $A \Rightarrow A$, (Ref)

together the following inference rules:

Composition: From $A \Rightarrow B$ and $A \Rightarrow C$ infer $A \Rightarrow B \oplus C$, (Comp)

Simplification: From $A \Rightarrow B$ and $C \Rightarrow D$ infer $A \oplus (C \ominus B) \Rightarrow D$, (Simp)

Extension: From $A \Rightarrow B$ infer $\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)$. (Ext)

SL for graded attribute implications

Derived rules

For all $A, B, C, D \in L^Y$ and $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$, the inference system consists of following axiom scheme:

$$\textit{Generalized Reflexivity}: \vdash A \Rightarrow B \text{ when } B \subseteq A \qquad (\texttt{GRef})$$

$$\textit{Transitivity}: A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C \qquad (\texttt{Tran})$$

$$\textit{Generalization}: A \Rightarrow B \vdash C \Rightarrow D \text{ when } A \subseteq C \text{ and } D \subseteq B \qquad (\texttt{Gen})$$

$$\textit{Generalized Composition}: A \Rightarrow B, C \Rightarrow D \vdash A \cup C \Rightarrow B \oplus D \qquad (\texttt{GComp})$$

$$\textit{Augmentation}: A \Rightarrow B \vdash A \cup C \Rightarrow B \oplus C \qquad (\texttt{Augm})$$

$$\textit{Generalized Transitivity}: A \Rightarrow B, B \cup C \Rightarrow D \vdash A \cup C \Rightarrow D \qquad (\texttt{GTran})$$

SL for graded attribute implications

Equivalences

The following equivalences hold:

$$\textit{Decomposition}: \{A \Rightarrow B\} \equiv \{A \Rightarrow B \ominus A\} \qquad (\texttt{DeEq})$$

$$\textit{Composition}: \{A \Rightarrow B, A \Rightarrow C\} \equiv \{A \Rightarrow B \oplus C\} \qquad (\texttt{CoEq})$$

$$\textit{Simplification}: \text{if } A \subseteq C, \{A \Rightarrow B, C \Rightarrow D\} \equiv \{A \Rightarrow B, C \ominus B \Rightarrow D \ominus B\} \qquad (\texttt{CoEq})$$

A Generalization...

SL for Graded Attribute Implications with General Semantics

$\mathbf{L} = \langle L, \leq, \oplus, \ominus, 0, 1 \rangle$

$\mathbb{S}$ be a $\mathbf{L}$-parameterization

$\mathbf{L}$-fuzzy sets $\mathbb{S}$-additives

Reflexivity: Infer $A \Rightarrow A$,

together the following inference rules:

Composition: From $A \Rightarrow B$ and $A \Rightarrow C$ infer $A \Rightarrow B \oplus C$,

Simplification: From $A \Rightarrow B$ and $C \Rightarrow D$ infer $A \oplus (C\text{-}B) \Rightarrow D$,

Extension: From $A \Rightarrow B$ infer $\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)$.

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Simplification: From $A \Rightarrow B$ and $C \Rightarrow D$ infer $A \oplus (C\text{-}B) \Rightarrow D$,

Extension: From $A \Rightarrow B$ infer $f(A) \Rightarrow f(B)$.

Consider $\oplus = \vee$

Given a hedge *

$\mathbb{L}$ – *parameterization* S

$\langle f_{c^*\otimes}, g_{c^*\to}\rangle$

$(f_{c^*\otimes}(A))(y) = c^* \otimes A(y)$

$(g_{c^*\to}(A))(y) = c^* \to A(y)$

A Generalization...

SL for Graded Attribute Implications with General Semantics

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S be a $\mathbb{L}$-parameterization

$\mathbb{L}$-fuzzy sets S-additives

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Given a hedge *

$\mathbb{L}$ – *parameterization* S

$\langle f_{c^*\otimes}, g_{c^*\rightarrow}\rangle$

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A Generalization...

SL for Graded Attribute Implications with General Semantics

$\mathbf{L} = \langle L, \leq, \oplus, \ominus, 0, 1\rangle$

S be a $\mathbb{L}$-parameterization

$\mathbb{L}$-fuzzy sets S-additives

Consider $\oplus =$

Consider $\ominus =$

$\mathbb{L} - parameterization\ S$

$f(A)(y) = \ldots$

$g(A)(y) = \ldots$

Reflexivity: Infer $A \Rightarrow A$,

together the following inference rules:

Composition: From $A \Rightarrow B$ and $A \Rightarrow C$ infer $A \Rightarrow B \oplus C$,

Simplification: From $A \Rightarrow B$ and $C \Rightarrow D$ infer $A \oplus (C\text{-}B) \Rightarrow D$,

Extension: From $A \Rightarrow B$ infer $\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)$.

LOGIC 1

LOGIC 2

LOGIC 3


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