Formal Equivalences Analysis

Francisco J. Valverde-Albacete and Carmen Peláez-Moreno and Pablo Cordero-Ortega and Inma de las Peñas and Manuel Ojeda-Aciego

[2]Dep. Matemática Aplicada Universidad de Málaga

1-3/11/2018, Jimena de la Fra.

Outline

1. Motivación
2. Plan de Ataque
3. Partition algebra
4. Towards Formal Equivalence Analysis

Outline

1 Motivación

2 Plan de Ataque

3 Partition algebra

4 Towards Formal Equivalence Analysis

Consideraciones Iniciales...

En un interesante “final remarks” de su artículo seminal sobre FCA, Wille renunció a cualquier atisbo de exhaustividad en su propuesta de reestructuración de la teoría de retículos y recomendó:

“Besides the interpretation by hierarchies of concepts, other basic interpretations of lattices should be introduced; ...”[a]

[a] Rudolf Wille. “Restructuring lattice theory: an approach based on hierarchies of concepts”. In: *Ordered sets (Banff, Alta., 1981).* Reidel, 1982, pp. 445–470.

Con esta perspectiva, debieramos considerar...

- Qué otra información porta un contexto formal.
- Qué pueden ser conceptualizaciones alternativas de esa información.

Consideraciones Iniciales. . .

En un interesante “final remarks” de su artículo seminal sobre FCA, Wille renunció a cualquier atisbo de exhaustividad en su propuesta de reestructuración de la teoría de retículos y recomendó:

“Besides the interpretation by hierarchies of concepts, other basic interpretations of lattices should be introduced; . . . ”[a]

[a] Rudolf Wille. “Restructuring lattice theory: an approach based on hierarchies of concepts”. In: *Ordered sets (Banff, Alta., 1981)*. Reidel, 1982, pp. 445–470.

Con esta perspectiva, debieramos considerar. . .

- Qué otra información porta un contexto formal.
- Qué pueden ser conceptualizaciones alternativas de esa información.

El Teorema Fundamental del FCA, Extendido (1/3)

Theorem

Sea el contexto formal (G, M, R) en donde,

- *G es un conjunto de objetos formales,*
- *M es un conjunto de attributos formales,*
- *$I \in 2^{g \times m}$ es una relación de incidencia.*

Entonces

(a) *La fase de análisis del contexto.*

  (i) *The polar operators $\cdot^{\uparrow} : 2^G \to 2^M$ and $\cdot^{\downarrow} : 2^M \to 2^G$.*

$$A^{\uparrow} = \{m \in M \mid \forall g \in A, gIm\} \qquad B^{\downarrow} = \{g \in G \mid \forall m \in B, gIm\}$$

form a Galois connection $(\cdot^{\uparrow}, \cdot^{\downarrow}) : 2^G \rightleftarrows 2^G$ whose formal concepts are the pairs (A, B) of closed elements such that $A^{\uparrow} = B \Leftrightarrow A = B^{\downarrow}$ whence

$$\mathfrak{B}(G, M, I) = \{(A, B) \in 2^G \times 2^M \mid A^{\uparrow} = B \Leftrightarrow A = B^{\downarrow}\}$$

El Teorema Fundamental del FCA, Extendido (2/3)

Theorem

(a) *Concepts are partially ordered with the hierarchical order as*

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2 .$$

and the set of formal concepts with the hierarchical order $\langle\mathfrak{B}(G,M,I),\leq\rangle$ *is a complete lattice* $\underline{\mathfrak{B}}(G,M,I)$ *called the concept lattice of* (G,M,I).

(b) *In* $\underline{\mathfrak{B}}(G,M,I)$ *infima and suprema are given by:*

$$\bigwedge_{t\in T}(A_t,B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)^{\uparrow}\right) \quad \bigvee_{t\in T}(A_t,B_t) = \left(\left(\bigcup_{t\in T} A_t\right)^{\downarrow}, \bigcap_{t\in T} B_t\right)$$

(c) *The mappings* $\overline{\gamma} : G \to V$ *and* $\overline{\mu} : M \to V$

$$g \mapsto \overline{\gamma}(g) = (\{g\}^{\uparrow\downarrow}, \{g\}^{\uparrow}) \qquad m \mapsto \overline{\mu}(m) = (\{m\}^{\downarrow}, \{m\}^{\downarrow\uparrow})$$

are such that $\overline{\gamma}(G)$ *is supremum-dense in* $\underline{\mathfrak{B}}(G,M,I)$, $\overline{\mu}(M)$ *is*

El Teorema Fundamental del FCA, Extendido (3/3)

Theorem

(b) *La fase de síntesis del contexto*

(i) *A complete lattice $\mathscr{V} = \langle V, \leq \rangle$ is isomorphic to (read can be built as) $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\overline{\gamma} : G \to V$ and $\overline{\mu} : M \to V$ such that*

- *$\overline{\gamma}(G)$ is supremum-dense in $\mathscr{V}$, $\overline{\mu}(M)$ is infimum-dense in $\mathscr{V}$, and*
- *gIm is equivalent to $\overline{\gamma}(g) \leq \overline{\mu}(m)$ for all $g \in G$ and all $m \in M$.*

(ii) *In particular, $\mathscr{V} \cong \underline{\mathfrak{B}}(V, V, \leq)$ using the assignments $G := J(\mathscr{V})$ and $M := M(\mathscr{V})$ where these are the sets of join- and meet-irreducibles, respectively, of $\mathscr{V}$.*

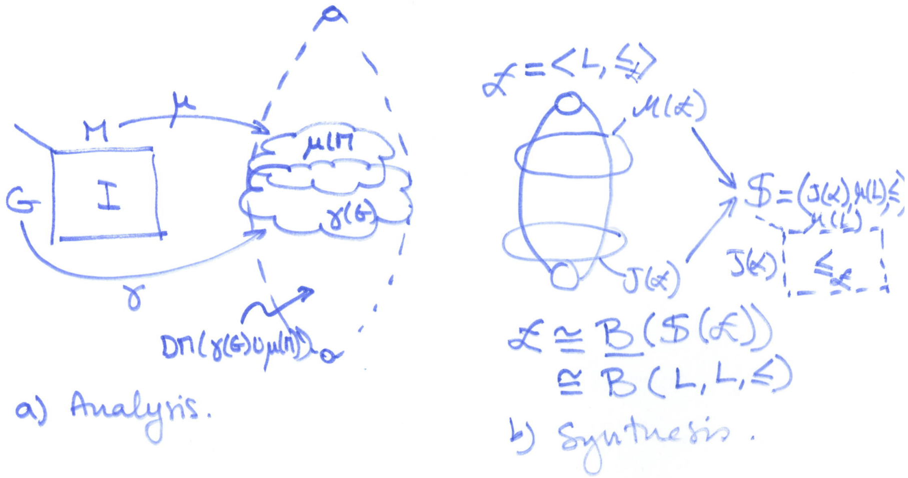


Figure: Las dos fases de FCA como una Transformada de Contextos: a) Análisis y b) Síntesis.

FCA tiene “Puntos Ciegos”

Los polares del contexto generan relaciones de equivalencia

Usando las funciones generadoras de contexto:

$$(g_1, g_2) \in \ker \overline{\gamma} \iff \overline{\gamma}(g_1) = \overline{\gamma}(g_2)$$
$$(m_1, m_2) \in \ker \overline{\mu} \iff \overline{\mu}(g_1) = \overline{\mu}(g_2)$$

El contexto purificado o reducido resintetizado por la parte de síntesis del Teorema Fundamental es

$$\mathbb{S}(\mathfrak{B}(G, M, I)) = (G/\ker \overline{\gamma}, M/\ker \overline{\mu}, I')$$

donde $([g]_{\ker \overline{\gamma}}, [m]_{\ker \overline{\mu}}) \in I' \iff gIm$.

Los elementos individuales de tales clases no están contenidos en $\mathscr{V}$ y no se pueden recuperar.

FCA tiene un “punto ciego” en
las clases de equivalencia de los polares.

FCA tiene “Puntos Ciegos”

Los polares del contexto generan relaciones de equivalencia

Usando las funciones generadoras de contexto:

$$(g_1, g_2) \in \ker \overline{\gamma} \iff \overline{\gamma}(g_1) = \overline{\gamma}(g_2)$$
$$(m_1, m_2) \in \ker \overline{\mu} \iff \overline{\mu}(g_1) = \overline{\mu}(g_2)$$

El contexto purificado o reducido resintetizado por la parte de síntesis del Teorema Fundamental es

$$\mathbb{S}(\mathfrak{B}(G, M, I)) = (G/\ker \overline{\gamma}, M/\ker \overline{\mu}, I')$$

donde $([g]_{\ker \overline{\gamma}}, [m]_{\ker \overline{\mu}}) \in I' \iff gIm$.

Los elementos individuales de tales clases no están contenidos en $\mathscr{V}$ y no se pueden recuperar.

FCA tiene un “punto ciego” en

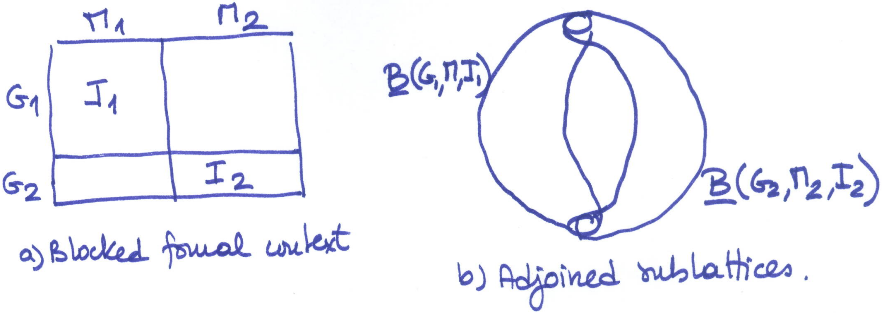
las clases de equivalencia de los polares.

FCA tiene “Espejismos” (I)

Los contextos formales diagonales por bloques son “especiales”.

- Cada bloque es independiente de los demás.
- Lo esperable es que cada bloque genere su propio retículo conceptual.

Pero FCA fuerza a que los retículos compartan $\top$ y $\bot$.

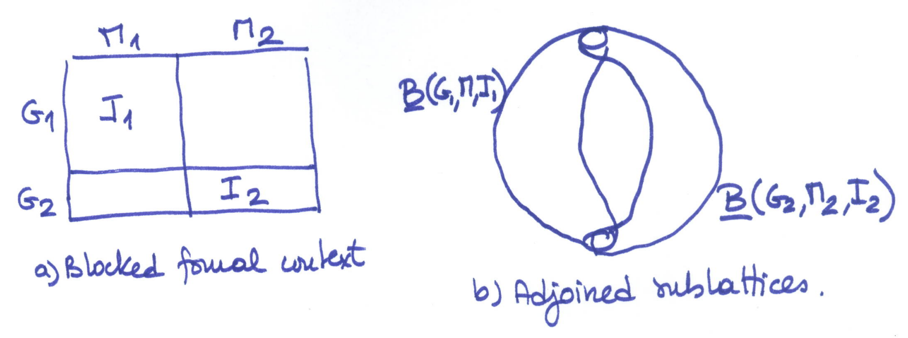


FCA tiene “Espejismos” (I)

Los contextos formales diagonales por bloques son “especiales”.

- Cada bloque es independiente de los demás.
- Lo esperable es que cada bloque genere su propio retículo conceptual.

Pero FCA fuerza a que los retículos compartan $\top$ y $\bot$.



En resumen. . .

FCA no extrae *toda* la información de un contexto formal.

- FCA ignora las equivalencias entre objetos y atributos inducidas por los polares.

FCA introduce cierta "información visual" en el retículo.

- FCA fuerza relaciones jerárquicas donde no las hay.

Necesitamos otras "lentes" para mirar en los "puntos ciegos" y deshacer los "espejismos" del FCA sobre un contexto formal.

Formal XX Analysis

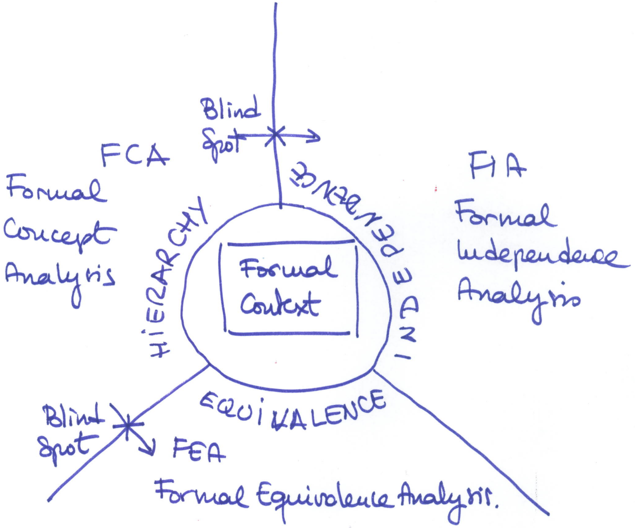


Figure: A variety of formal analysis around a formal context are conceivable: formal concept analysis around the notion of hierarchy as captured by upper

Outline

1. Motivación
2. Plan de Ataque
3. Partition algebra
4. Towards Formal Equivalence Analysis

Componentes de un “Formal X Analysis”

Si analizamos el Teorema Fundamental, vemos que hay ciertos “ingredientes” fundamentales.

- Una conexión (adjunción) de Galois entre los conjuntos potencia de objetos y atributos, de forma que los pares de subconjuntos estables capturen un “sabor” del retículo.
  - FCA => Los conjuntos de cotas sup. e inf. capturan “jerarquía”
- Un “gránulo” que captura nuclearmente dicho “sabor”.
  - FCA => Supremo- e ínfimo-irreducibles.
- Un par de funciones(no inyectivas!) de los objetos formales y los atributos en los gránulos:
  - FCA => $\bar{\gamma}$ y $\bar{\mu}$ generadores de conceptos.
- Un teorema de representación en función de los “gránulos” de determinada capacidad(universal, lo mejor).
  - FCA => Teorema de Dedekind-McNeille $\mathscr{L} \equiv DM(\mathscr{J}(\mathscr{L}) \cup \mathscr{M}(\mathscr{L}))$

Ejemplo: Formal Independence Analysis

Sea un orden parcial finito $\mathscr{P} = \langle P, \leq \rangle$

- Las anticadenas capturan la noción de no-relación.
- No conocemos un teorema de representación pero existe...

Theorem (Behrendt's representation theorem)

Let $\mathscr{L} = \langle L, \leq \rangle$ be a lattice. Then there exists a poset $\mathscr{P} = \langle P, \leq_P \rangle$ such that $|P| = 2|L|$, where any chain has at most 2 elements and such that $\mathscr{L}$ is isomorphic to the lattice of maximal antichains of $(P, \leq_P)$.

$$\mathscr{L} \cong MA(\mathscr{P})$$

- Esto sugiere que las anticadenas maximales (y minimales para el dual) son los "gránulos".
- Existen conexiones de Galois entre los conjuntos de cotas superiores e inferiores y las anticadenas maximales y minimales.

El Teorema Fundamental del FIA (1/3)

Theorem (Basic theorem of formal independence analysis (Valverde, Peláez, Cabrera, Cordero, Ojeda, 2018))

(a) *The context analysis phase: Given a formal context (G, M, I),*

(i) *The context operators $\cdot^{\sim} : 2^G \to 2^M$ and $\cdot_{\sim} : 2^M \to 2^G$*

$$\alpha^{\sim} = M \setminus \uparrow \alpha \qquad\qquad \beta_{\sim} = G \setminus \downarrow \beta$$

form a right-Galois connection $(\cdot^{\sim}, \cdot_{\sim}) : (2^G, \subseteq) \leftharpoonup (2^M, \subseteq)$ whose formal tomoi are the pairs (α, β) such that $\alpha^{\sim} = \beta$ and $\alpha = \beta_{\sim}$.

(ii) *The set of formal tomoi $\mathfrak{A}(G, M, I)$ with the relation*

$$(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \ \text{ iff } \ \alpha_1 \supseteq \alpha_2 \ \text{ iff } \ \beta_1 \subseteq \beta_2$$

is a complete lattice, which is called the tomoi lattice of (G, M, I) and denoted $\underline{\mathfrak{A}}(G, M, I)$, where infima and suprema are given by:

$$\bigwedge_{t \in T} (\alpha_t, \beta_t) = \Big(\bigcup_{t \in T} \alpha_t, \Big(\bigcap_{t \in T} \beta_t \Big)_{\sim}^{\ \sim} \Big) \qquad \bigvee_{t \in T} (\alpha_t, \beta_t) = \Big(\Big(\bigcap_{t \in T} \alpha_t \Big)^{\sim}_{\ \sim}, \bigcup_{t \in T} \beta_t \Big)$$

El Teorema Fundamental del FIA (1/3)

Theorem

(a) *The mappings $\overline{\gamma}: G \to \underline{\mathfrak{A}}(G,M,I)$ and $\overline{\mu}: M \to \underline{\mathfrak{A}}(G,M,I)$*

$$g \mapsto \overline{\gamma}(g) = (\{g\}^{\sim}{}_{\sim}, \{g\}^{\sim}) \qquad m \mapsto \overline{\mu}(m) = (\{m\}_{\sim}, \{m\}_{\sim}{}^{\sim})$$

are such that $\overline{\gamma}(G)$ is infimum-dense in $\underline{\mathfrak{A}}(G,M,I)$, $\overline{\mu}(M)$ is supremum-dense in $\underline{\mathfrak{A}}(G,M,I)$.

(b) *The context synthesis phase: Given a complete lattice $\mathbb{L} = \langle L, \leq \rangle$*

(i) *$\mathbb{L}$ is isomorphic to[a] $\underline{\mathfrak{A}}(G,M,I)$ if and only if there are mappings $\overline{\gamma}: G \to L$ and $\overline{\mu}: M \to L$ such that*

- *$\overline{\gamma}(G)$ is infimum-dense in $\mathbb{L}$, $\overline{\mu}(M)$ is supremum-dense in $\mathbb{L}$, and*
- *$g\,I\,m$ is equivalent to $\overline{\gamma}(g) \not\geq \overline{\mu}(m)$ for all $g \in G$ and all $m \in M$.*

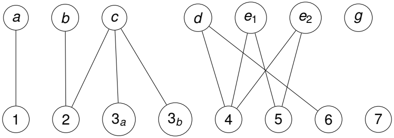
(ii) *In particular, $\mathbb{L} \cong \underline{\mathfrak{A}}(L, L, \not\geq)$ and, if L is finite, $\mathbb{L} \cong \underline{\mathfrak{A}}(M(\mathbb{L}), J(\mathbb{L}), \not\geq)$ where $M(\mathbb{L})$ and $J(\mathbb{L})$ are the sets of meet- and join-irreducibles, respectively, of $\mathbb{L}$.*

[a] Read *can be built as*.

Example context

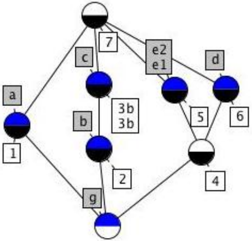
$\mathbb{K}_1$	a	b	c	d	e1	e2	g
1	×						
2		×	×				
3a			×				
3b			×				
4				×	×	×	
5					×	×	
6				×			
7							

(a) Tabular representation of $\mathbb{K}_1$

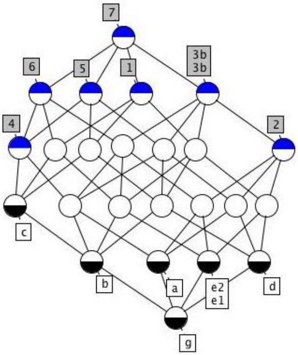


(b) Bipartite graph representation of $\mathbb{K}_1$

Figure: **Equivalent representations of an example context $\mathbb{K}_1 = (G, M, I)$.** (a)) tabular representation. (b)) bipartite graph representation.



(a) Concept lattice $\mathfrak{B}(G, M, I)$



(b) Tomoi lattice $\mathfrak{A}(G, M, I)$

Figure: **Two different lattices for context** $\mathbb{K}_1 = (G, M, I)$ (a) lattice of formal concepts *showing* three adjoint sublattices, and (b) lattice of formal tomoi, *describing* the three adjoint sublattices.

Outline

1 Motivación

2 Plan de Ataque

3 Partition algebra

4 Towards Formal Equivalence Analysis

Carrying formal analysis on partitions

We have seen that:

- There are partitions $\ker \overline{\gamma}$ and $\ker \overline{\mu}$ related to a formal context.
- The FCA procedure is blind to the classes in these partitions, that is, there is an information loss in the process of FCA.
- There is a notion of "mutual determination" of objects and attributes, whether from the qualitative point of view of Psychology and Cognition, or from the purely quantitative of Information Theory.

Define the *coarsening order*:

Let $\mathrm{Part}(G)$ denote the set of partitions over G.

$$\pi \leq \sigma \iff \forall g_1, g_2 \in G, g_1 \equiv g_2(\pi) \text{ implies that } g_1 \equiv g_2(\sigma)$$

The lattice of partitions of a set

Theorem

Let G be a set. Then $Part(G) = \langle \Pi(G), \subseteq \rangle$ is a complete lattice, called the partition lattice (or equivalence lattice of G) where:

- *The bottom of $Part(G)$ is $\iota_G = \{\{g\} \mid g \in G\}$ is the set of trivial blocks.*
- *The top of $Part(G)$ is $\omega_G = \{G\}$.*
- *The meet of partitions $\{\pi_i \mid i \in I\}$ is defined, for all $g_1, g_2 \in G$ as:*

$$g_1 \equiv g_2(\wedge_{i \in I} \pi_i) \iff \forall i \in I, g_1 \equiv g_2(\pi_i)$$

- *The join of partitions $\{\pi_i \mid i \in I\}$ is defined, for all $g_1, g_2 \in G$ as:*

$$g \equiv d(\vee_{i \in I} \pi_i) \iff$$

there is a natural number n, a subset $J = \{i_0, \ldots, i_n\} \subseteq I$, and $g_0, \ldots, g_{n+1} \in G$ such that $g = g_0, \ldots, d = g_{n+1}$ and $g_j = g_{j+1}(\pi_{i_j})$, for $0 \leq j \leq n$

More partitions

Theorem (Continued)

- *The atoms of* $Part(G)$ *are the partitions with exactly one non-trivial block and this block has two elements.*
- *The co-atoms of* $Part(G)$ *are the partitions with exactly two blocks.*
- *The covering relation in* $Part(G)$ *holds* $\pi \prec \sigma$ *iff* σ *is the result of replacing two blocks of* π *by their union.*

Proposition

Let G *be a set and consider* $\pi \in \Pi(G)$*. Then,*

- $\uparrow \pi \subseteq Part(G)$ *is isomorphic to the partition lattice of the set* π*,* $\uparrow \pi \cong Part(\pi)$*.*
- $\downarrow \pi \subseteq Part(G)$ *is isomorphic to direct product of* $Part(X)$ *where* X *ranges over the non-trivial blocks of* π*,* $\downarrow \pi \cong \times_{X \text{ non trivial in } \pi} Part(X)$*.*

La “bombilla” de las particiones

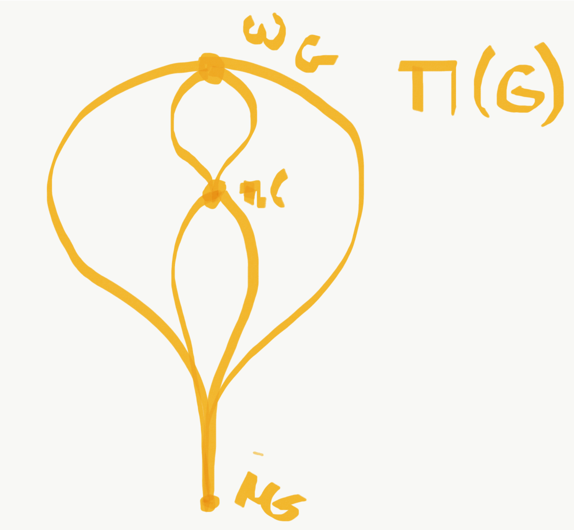


Figure: Sketch of the lattice of partitions with the filter and ideal of $\pi \in \text{Part}(G)$ drawn.

Outline

1 Motivación

2 Plan de Ataque

3 Partition algebra

4 Towards Formal Equivalence Analysis

The Analysis Part

The "reduced" formal context

Let $\mathbb{K} = (G, M, I)$ be a formal context between a set of objects G and attributes M related by $I \in G \times M$.

$$(g_1, g_2) \in \ker \overline{\gamma} \iff \overline{\gamma}(g_1) = \overline{\gamma}(g_2) \quad (m_1, m_2) \in \ker \overline{\mu} \iff \overline{\mu}(g_1) = \overline{\mu}(g_2) \tag{1}$$

and define the *reduced* [2] or *purified* [1] formal context $\mathbb{K}_0 = (G_0, M_0, I_0)$ where $G_0 = G/\ker \overline{\gamma}$, $M_0 = M/\ker \overline{\mu}$ and $([g]_{\ker \overline{\gamma}}, [m]_{\ker \overline{\mu}}) \in I_0 \iff gIm$.

We already have two "lattices of partitions" induced.

$$\text{Part}\,(\ker \overline{\gamma}) \cong \uparrow \ker \overline{\gamma} \subseteq \Pi(G) \qquad \text{Part}\,(\ker \overline{\mu}) \cong \uparrow \ker \overline{\mu} \subseteq \Pi(M) \tag{2}$$

Looking for meet irreducibles

Assumption

The minimal distinction of objects in terms of their attributes must come from the partitions $\pi(m) = \{m^{\downarrow}, G \setminus m^{\downarrow}\}$.

$$\forall m \in M, \pi(m) = \{([m]_{\ker \overline{\mu}})^{\downarrow}, G \setminus ([m]_{\ker \overline{\mu}})^{\downarrow}\} \quad (3)$$

- The dual is not so important cognitively speaking:

$$\forall g \in G, \sigma(g) = \{([g]_{\ker \overline{\gamma}})^{\uparrow}, M \setminus ([g]_{\ker \overline{\gamma}})^{\uparrow}\} \quad (4)$$

- There are at most $|\ker \overline{\gamma}|$ and $|\ker \overline{\mu}|$ of those partitions.
- Both types of partitions are co-atomic in $\Pi(M)$ and $\Pi(G)$,
- respectively.

$$\ker \overline{\gamma} \leq \bigwedge_{m \in M} \pi(m) \qquad \ker \overline{\mu} \leq \bigwedge_{g \in G} \sigma(g) \quad (5)$$

Looking for joint irreducibles

Informal reasoning:

- The easiest partition to work with using joins are the identities ι_G and ι_M since they are the join null elements.
- If the only non-trivial block of σ is $B \subset M$, then the effect of joining $\sigma' \vee \sigma$ is to collapse all of the classes that have elements of B.
- So in general for all $B \subseteq G$, we define $\sigma(B) = \{B\} \cup \iota_{M \setminus B}$ as a partition that is "suitably" low in $\Pi(M)$.
- The object-induced attribute partitions that we propose as elements of the join-dense set are:

$$\sigma(g) = \{g^{\uparrow}, \iota_{M \setminus g^{\uparrow}}\} \tag{6}$$

The type of connection

By the ideas exposed above, we are aiming at building a left adjunction over partitions of objects and attributes (required by the analysis part of the theorem) $(\cdot^*, \cdot_*) : \Pi(G) \leftrightharpoons \Pi(M)$. The idea is:

- to have a left adjoint that is a join morphism, so that the joins of object partitions can be carried out as the (attribute) joins of their images (with easy-to-calculate images, of course)

$$(\pi_1 \vee \pi_2)^* = (\pi_1)^* \vee (\pi_2)^* \tag{7}$$

- to have a right adjoint that is a meet morphism, so that the meets of attribute partitions can be carried out as the (object) meets of their images.

$$(\sigma_1 \wedge \sigma_2)_* = (\sigma_1)_* \wedge (\sigma_2)_* \tag{8}$$

The Synthesis Part

Theorem (Whitman [whi:46])

Every lattice can be embedded in some partition lattice.

For now, suppose that we find an algorithm to obtain a lattice of partitions related to a context $\mathbb{K}$ and we denote the resulting lattice by $\mathfrak{P}(\mathbb{K})$.

The synthesis part of the Fundamental Theorem of FEA would be:

Lemma (To be proven!)

Given L, there is a context $\mathbb{K}(L)$, systematically obtainable from L, whose partition lattice is isomorphic to it $L \cong \mathfrak{P}(\mathbb{K}(L))$

A picture is worth a thousand words

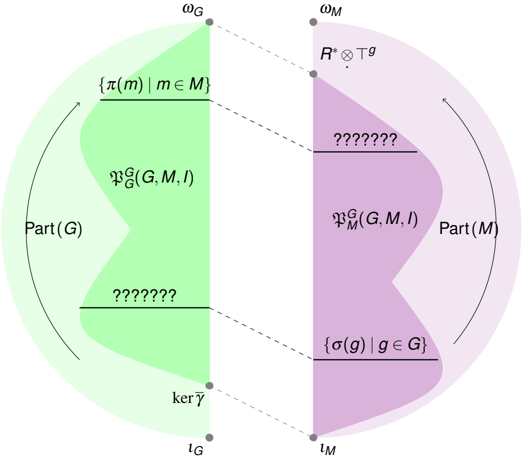


Figure: Intended schematics of the adjunction $\mathfrak{P}^G(G, M, I)$ between Part (G) and Part (M). Content-induced partitions $\mathfrak{P}^G_G(G, M, I)$ and span-induced partitions $\mathfrak{P}^G_M(G, M, I)$ are isomorphic.

A preliminary example

	a	b	c	d	e	f
1	×				×	
2		×	×		×	×
3a			×		×	
3b			×		×	

(a) (G, M, I).

$$R_I = \begin{bmatrix} \top & \cdot & \cdot & \cdot & \top & \cdot \\ \cdot & \top & \top & \cdot & \top & \top \\ \cdot & \cdot & \top & \cdot & \top & \cdot \\ \cdot & \cdot & \top & \cdot & \top & \cdot \end{bmatrix}$$

(b) (G, M, R_I)

Figure: A context and an encoding R_I of the incidence I into a complete semifield $\overline{\mathscr{K}}$ (see text).

Example: Object and attribute "concepts" of adjunction

$$\Gamma_C = \begin{bmatrix} \top & \cdot & \cdot & \cdot \\ \cdot & \top & \top & \top \\ \cdot & \cdot & \top & \top \\ \cdot & \cdot & \top & \top \end{bmatrix} \qquad M_C = \begin{bmatrix} \top & \cdot & \cdot & \cdot & \top & \cdot \\ \cdot & \top & \top & \cdot & \top & \top \\ \cdot & \cdot & \top & \cdot & \top & \cdot \\ \cdot & \cdot & \top & \cdot & \top & \cdot \end{bmatrix}$$

$$\Gamma_S = \begin{bmatrix} \cdot & \top & \top & \top \\ \top & \cdot & \top & \top \\ \top & \cdot & \cdot & \cdot \\ \top & \top & \top & \top \\ \cdot & \cdot & \cdot & \cdot \\ \top & \cdot & \top & \top \end{bmatrix} \qquad M_S = \begin{bmatrix} \cdot & \top & \top & \cdot & \top & \top \\ \top & \cdot & \top & \cdot & \top & \cdot \\ \top & \cdot & \cdot & \cdot & \top & \cdot \\ \top & \top & \top & \top & \top & \top \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \top & \cdot & \top & \cdot & \top & \cdot \end{bmatrix}$$

$1 \quad 2 \quad 3a \quad 3b \qquad\qquad a \quad b \quad c \quad d \quad e \quad f$

The attribute-induced object partitions

The object partitions induced by the attributes:

$$\pi(a) = \{\{1\}, \{2, 3a, 3b\}\} = \pi(c)$$
$$\pi(b) = \{\{2\}, \{1, 3a, 3b\}\} = \pi(f)$$
$$\pi(d) = \{G, \varnothing\} = \pi(e)$$

- the bottom is the partition $\ker \overline{\gamma} = \{\{1\}, \{2\}, \{3a, 3b\}\}$.
- we have had to consider the empty set to be included in the partition to make $\pi(d) = \pi(e)$.

The object-induced attribute partitions

- The attribute partitions induced by the objects:

$$\sigma(1) = \{\{b,c,d,f\},\{a\},\{e\}\}$$
$$\sigma(2) = \{\{a,d\},\{b\},\{c\},\{e\},\{f\}\}$$
$$\sigma(3a) = \{\{a,b,d,f\},\{c\},\{e\}\} = \sigma(3b)$$

- with bottom and top:

$$\iota_M = \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\} \quad \omega_M = \{\{a,b,c,d,f\},\{e\}\}$$

Conclusiones. . .

¡Gracias!
¿Alguna pregunta?
¿Alguna sugerencia?