Reducing concept lattices by means of a weaker notion of congruence

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Introduction

- Formal Concept Analysis (FCA) is a theory of data analysis which identifies conceptual structures among data sets.
- One of the most appealing topics within FCA is related to reduction mechanisms preserving the main information of the database.
- Recently, a mechanism to reduce formal contexts based on the philosophy of reduction considered in Rough Set Theory was presented in:
  - M. J. Benítez-Caballero, J. Medina, E. Ramírez-Poussa and D. Ślęzak. A computational procedure for variable selection preserving different initial conditions. International Journal of Computer Mathematics, 0(0):1–18, 2019.
- We are interested in complementing the research carried out in the previously mentioned paper.

Introduction

- In particular, in the study of mechanisms to reduce concept lattices by means of equivalence classes, which are convex sublattices of the original lattice, in the concepts.
- This target is achieved by the notion of congruence relation on lattices, but it leads to a significant loss of information.
- Therefore, we introduce a weaker notion of congruence that improves the results provided by congruence relations.
- In addition, we present some results and features from this new notion.

Results from attribute reduction mechanism in FCA

Proposition

Given a context (A, B, R) *and a subset* $D \subseteq A$*. The set* $R_E = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ *is an equivalence relation. Where* ${}^{\uparrow_D}$ *denotes the concept-forming operator restricted to the subset of attributes* $D \subseteq A$*.*

Proposition

Given a context (A, B, R)*, a subset* $D \subseteq A$ *and a class* $[(X, Y)]_D$ *of the quotient set* $\mathcal{C}(A, B, R)/R_E$*. The class* $[(X, Y)]_D$ *is a join semilattice with maximum element* $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$*.*

Congruences

We introduce the notion of congruence on a lattice. We write $(a, b) \in \theta$ to indicate that a and b are related under the equivalence relation θ.

Definition

We say that an equivalence relation θ on a given lattice $(L, \preceq)$ is *compatible* with the supremum and the infimum of $(L, \preceq)$ if, for all $a, b, c, d \in L$,
$(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a \vee c, b \vee d) \in \theta$ and $(a \wedge c, b \wedge d) \in \theta$.

Definition

Given a lattice $(L, \preceq)$, we say that an equivalence relation on L, which is compatible with both the supremum and the infimum of $(L, \preceq)$ is a *congruence* on L.

The quotient lattice from a congruence

We introduce the notion of quotient lattice from a congruence based on the operations of the original lattice.

Definition

Given an equivalence relation θ on a lattice L, two operators $\vee_\theta$ and $\wedge_\theta$ on the set of equivalence classes $L/\theta = \{[a]_\theta \mid a \in L\}$, for all $a, b \in L$, are defined as follows

$$[a]_\theta \vee_\theta [b]_\theta = [a \vee b]_\theta \text{ and } [a]_\theta \wedge_\theta [b]_\theta = [a \wedge b]_\theta .$$

$\vee_\theta$ and $\wedge_\theta$ are well defined on L/θ if and only if θ is a congruence. When θ is a congruence on L, we call $\langle L/\theta, \vee_\theta, \wedge_\theta \rangle$ *the quotient lattice of* L *modulo* θ.

Notions about congruences

- Given a lattice L and a subset of L composed of four elements $\{a, b, c, d\}$, if $a < b$, $c < d$ and either ($a \vee d = b$ and $a \wedge d = c$) or ($b \vee c = d$ and $b \wedge c = a$), then a, b and c, d are *opposite sides of the quadrilateral*.
- The equivalence classes of a congruence are *quadrilateral-closed* if whenever given two opposite sides of a quadrilateral a, b and c, d, where $a, b \in [x]_\theta$, with $x \in L$ then there exists $y \in L$ such that $c, d \in [y]_\theta$.

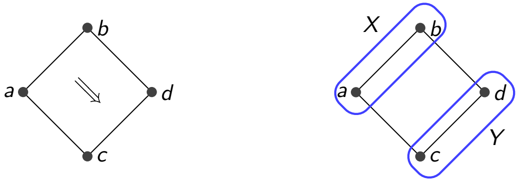


Figure: Opposite sides of a quadrilateral (left) and quadrilateral-closed (right).

Characterization of congruences

The following result shows a characterization of congruences in terms of their equivalence classes.

Theorem

Let $(L, \preceq)$ be a lattice and let θ be an equivalence relation on L. Then θ is a congruence if and only if:

(i) each equivalence class of θ is a sublattice of L,

(ii) each equivalence class of θ is convex,

(iii) the equivalence classes of θ are quadrilateral-closed.

Example

Formal context (A, B, R), where

B ={Mercury (M), Venus (V), Earth (E), Mars (Ma), Jupiter (J), Saturn (S), Uranus (U), Neptune (N), Pluto (P)},

A ={small size (ss), medium size (ms), large size (ls), near sun (ns), far sun, (fs), moon yes (my), moon no (mn)}

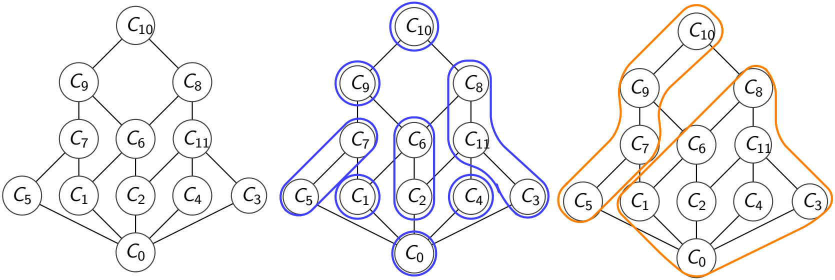
and R is given by the following table.

R	M	V	E	Ma	J	S	U	N	P
small size	1	1	1	1	0	0	0	0	1
medium size	0	0	0	0	0	0	1	1	0
large size	0	0	0	0	1	1	0	0	0
near sun	1	1	1	1	0	0	0	0	0
far sun	0	0	0	0	1	1	1	1	1
moon yes	0	0	1	1	1	1	1	1	1
moon no	1	1	0	0	0	0	0	0	0

Table: Relation R.

Example

The concept lattice obtained from the context (left), the equivalence classes which group concepts of the original concept lattice (middle), using the attribute reduction mechanism previously mentioned, and the smallest congruence containing such a reduction (right).



Weak-Congruence

The result obtained in the previous example highlights the necessity of a weaker notion of congruence.

Definition

Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a *weak-congruence* if and only if

(i) each equivalence class of δ is a sublattice of L,

(ii) each equivalence class of δ is convex.

Definition

Let $(L, \preceq)$ be a lattice and δ a weak-congruence, the quotient set L/δ provides a partition of L, which is called *weak-congruence partition* (or *wc-partition* in short) of L and it is denoted as π_δ. The elements in the wc-partition π_δ are convex sublattices of L.

Characterization of weak-congruences

Proposition

Given a lattice $(L, \preceq)$ and an equivalence relation δ on L, the relation δ is a weak-congruence on L if and only if, for each $a, b, c \in L$, the following properties hold:

(i) *If $(a, b) \in \delta$ and $a \preceq c \preceq b$, then $(a, c) \in \delta$.*

(ii) *$(a, b) \in \delta$ if and only if $(a \wedge b, a \vee b) \in \delta$.*

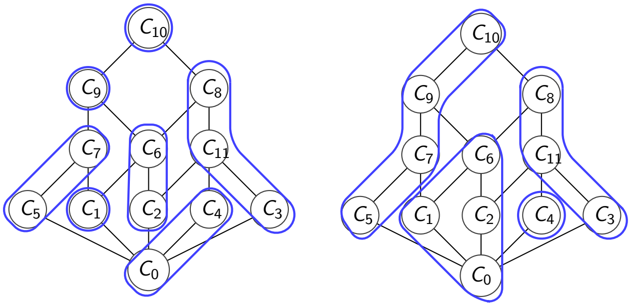
Weak-congruences can be ordered by using the definition of inclusion of equivalence relations, which is recalled in the following.

Definition

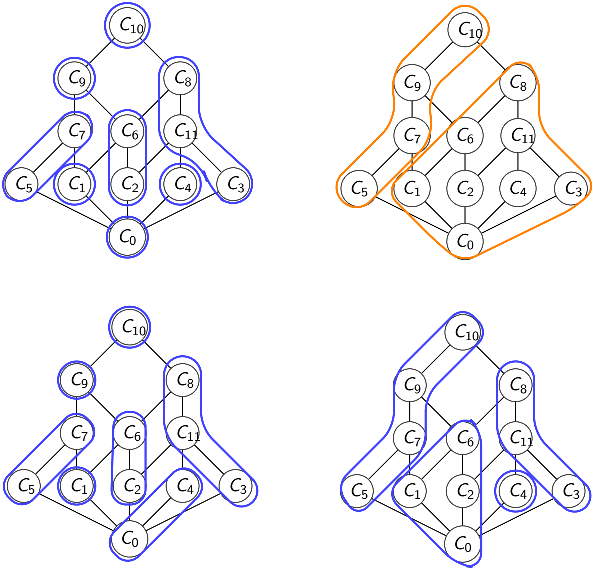
Let ρ_1 and ρ_2 be two equivalence relations on a lattice $(L, \preceq)$. We say that the equivalence relation ρ_1 is included in ρ_2, denoted as $\rho_1 \sqsubseteq \rho_2$, if for every equivalence class $[x]_{\rho_1} \in L/\rho_1$ there exists an equivalence class $[y]_{\rho_2} \in L/\rho_2$ such that $[x]_{\rho_1} \subseteq [y]_{\rho_2}$.
We say that two equivalence relations, ρ_1 and ρ_2, are incomparable if $\rho_1 \not\sqsubseteq \rho_2$ and $\rho_2 \not\sqsubseteq \rho_1$.

Example

There exist different weak-congruences containing the partition given by the reduction previously mentioned. Here, we show two of them which are incomparable and provide more granular partitions than the one given by the usual definition of congruence.



Example



Ordering equivalence classes of a weak-congruence

How can we establish an ordering relation between these classes?
The following definition will play a key role for this purpose.

Definition

Let $(L, \preceq)$ be a lattice and a weak-congruence δ on L.

(i) A sequence of elements of L, $\{p_0, p_1, \ldots, p_n\}$ with $n \geq 1$, is called a δ-*sequence*, denoted as $(p_0, p_n)_\delta$, if for each $i \in \{1, \ldots, n\}$ either $(p_{i-1}, p_i) \in \delta$ or $p_{i-1} \preceq p_i$ holds.

(ii) If a δ-sequence, $(p_0, p_n)_\delta$, satisfies that $p_0 = p_n$, then it is called a δ-*cycle*. In addition, if the δ-cycle satisfies that $[p_0]_\delta = [p_1]_\delta = \cdots = [p_n]_\delta$ is said to be *closed*.

Ordering equivalence classes of a weak-congruence

The previous notions are clarified in the following figure, where the triple vertical line that connects p_{i-1} with p_i means that $(p_{i-1}, p_i) \in \delta$ and the simple line indicates that the two elements are connected by means of the ordering relation defined on the lattice.

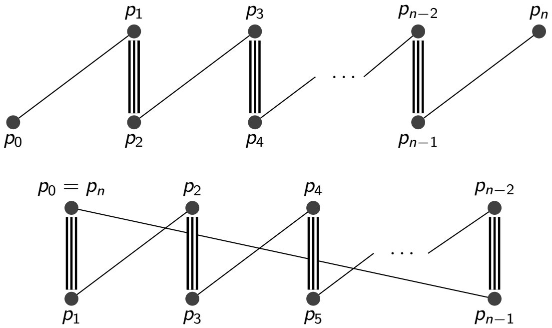


Figure: Example of δ-sequence (top) and δ-cycle (bottom).

Defining a preorder for a weak-congruence

The following definition provides a first step to define a partial order on the quotient set provided by a weak-congruence.

Definition

Given a lattice $(L, \preceq)$ and a weak-congruence δ on L, we define a binary relation $\preceq_\delta$ on L/δ as follows:

$$[x]_\delta \preceq_\delta [y]_\delta \quad \text{if there exists a } \delta\text{-sequence } (x', y')_\delta,$$

for some $x' \in [x]_\delta$ and $y' \in [y]_\delta$.

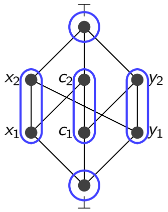


Figure: Example where $\preceq_\delta$ is not a partial order.

The relation $\preceq_\delta$ is a partial order

The following results state different conditions under which the relation $\preceq_\delta$ is a partial order.

Proposition

Given a lattice $(L, \preceq)$ *and a weak-congruence* δ*, if for any two equivalence classes* $[x]_\delta, [y]_\delta \in L/\delta$ *there exists only one class* $[c]_\delta \in L/\delta$ *such that* $[x]_\delta \preceq_\delta [c]_\delta \preceq_\delta [y]_\delta$ *and* $[y]_\delta \preceq_\delta [c]_\delta \preceq_\delta [x]_\delta$ *satisfying that* $x_1 \preceq c_1 \preceq y_1$ *and* $y_2 \preceq c_2 \preceq x_2$ *with* $x_1, x_2 \in [x]_\delta$*,* $c_1, c_2 \in [c]_\delta$ *and* $y_1, y_2 \in [y]_\delta$*, then* $[x]_\delta = [y]_\delta$*.*

Corollary

Given a lattice $(L, \preceq)$ *and a weak-congruence* δ*, if for any two equivalence classes* $[x]_\delta, [y]_\delta \in L/\delta$ *such that* $[x]_\delta \preceq_\delta [y]_\delta$ *and* $[y]_\delta \preceq_\delta [x]_\delta$ *satisfy that* $x_1 \preceq y_1$ *and* $y_2 \preceq x_2$ *with* $x_1, x_2 \in [x]_\delta$ *and* $y_1, y_2 \in [y]_\delta$*, then* $[x]_\delta = [y]_\delta$*.*

The relation $\preceq_\delta$ is a partial order

The relation $\preceq_\delta$ is not a partial order because of there may exist a δ-cycle composed of elements belonging to different equivalence classes. In order to avoid this problem, we state the next result.

Lemma

Given a lattice $(L, \preceq)$ and a weak-congruence δ on L, the preorder $\preceq_\delta$ given in the previous definition is a partial order if and only if every δ-cycle in L is closed.

Reduction mechanism of concept lattices

Procedure to reduce concept lattices by using weak-congruences:

1. We consider the concept lattice $\mathcal{C}(A, B, R)$ and the partition induced by a reduction of the set of attributes. We define the smallest weak-congruence δ that contains such a partition.
2. The preorder $\preceq_\delta$ is defined as in the previous Definition.
   - **2.1** If every δ-cycle is closed then we have that $\preceq_\delta$ is a partial order on $\mathcal{C}(A, B, R)/\delta$ by Lemma , and the procedure finishes.
   - **2.2** Otherwise, a new equivalence relation is defined as follows $\rho = \{([x]_\delta, [y]_\delta) \in L/\delta \times L/\delta \mid [x]_\delta \leq_\delta [y]_\delta$ and $[y]_\delta \leq_\delta [x]_\delta\}$, this implies that $\delta \sqsubseteq \rho$.
     - **(b1)** If ρ is a weak-congruence, the procedure ends.
     - **(b2)** Otherwise, we compute the smallest weak-congruence $\bar{\delta}$ such that $\delta \sqsubseteq \rho \sqsubseteq \bar{\delta}$. In this way, we have that $\preceq_{\bar{\delta}}$ is a partial order on $L/\bar{\delta}$ and the procedure ends.

Example

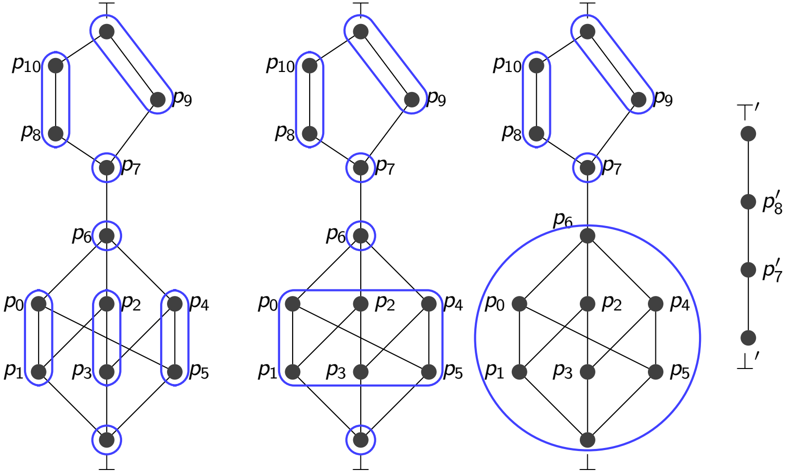


Figure: The lattice L and weak-congruence δ (left), the equivalence relation ρ on L/δ (middle-left), the least weak-congruence $\bar{\delta}$ containing ρ (middle-right) and the quotient set $L/\bar{\delta}$ (right).

Conclusions and future work

- In this work, we have considered one partition of a concept lattice in which the equivalence classes are join-semilattices and we have applied congruences in order to get a better algebraic structure.
- We have introduced a weaker definition of congruence which can be applied to the problem of attribute reduction in FCA offering a better reduction than the one provided using congruence.
- We have endowed the equivalence classes of a weak-congruence with a partial order and we propose a procedure to reduce concept lattices.
- As future work, this study will be continued including complementary properties and features related to weak-congruences.

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