A VIEW OF f-INDEXES OF INCLUSION UNDER DIFFERENT AXIOMATIC DEFINITIONS OF FUZZY INCLUSION

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Introduction

- The notion of fuzzy sets were introduced by Zadeh in 1965 as a convenient generalization of crisp (standard) sets to deal with uncertainty.
- Since such a moment, the extension of crisp set operations and relations has taken the attention of many researchers.
- However, despite of the time passed, there is not a consensus on how to extend some of them yet.
- *The inclusion* is an example of this fact.
- In the literature we can find two different kind of approaches:
  - constructive (i.e., there is a formula or method to measure the inclusion)
  - axiomatic (i.e., a proposal of some basic properties that must be satisfied by any inclusion measure)

Introduction

In this talk I

- recall the notion of f-inclusion, which represents a kind of inclusion of a fuzzy set into another,
- recall the f-index of inclusion, which is a representation of the inclusion of a fuzzy set into another.
- compare the f-index of inclusion with five axiomatic measures of inclusion usually considered in the literature, namely:
  - Sinha-Dougherty,
  - Kitainik,
  - Young and
  - Fan-Xie-Pei (x2)

Basic notions

Fuzzy sets

Definition

A fuzzy set A is a pair $(\mathcal{U}, \mu_A)$ where:

- $\mathcal{U}$ is a non empty set (universe of A) and
- μ_A is a mapping from $\mathcal{U}$ to $[0, 1]$ (membership function of A).

In general, the universe is a fixed set for all the fuzzy sets considered.

Therefore, each fuzzy sets is generally determined by its membership function.

Hence, for the sake of clarity, we identify fuzzy sets with membership functions (i.e., $A(u) = \mu_A(u)$).

Basic notions

Operations between fuzzy sets

The set of fuzzy sets defined on the universe $\mathcal{U}$ is denoted by $\mathcal{F}(\mathcal{U})$.

On $\mathcal{F}(\mathcal{U})$ we can extend the usual crisp operations of union, intersection and complement as follows.

Definition

Given two fuzzy sets A and B, we define

- (union) $A \cup B(u) = \max\{A(u), B(u)\}$
- (intersection) $A \cap B(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = n(A(u))$

where $n\colon [0,1] \to [0,1]$ is an involutive negation operator; i.e., n is a decreasing mapping such that $n(0) = 1$, $n(1) = 0$ and $n(n(x)) = x$ for all $x \in [0,1]$.

Basic notions

Definition

An implication $I\colon [0,1]\times[0,1]$ is any mapping decreasing in its first component, increasing in the second component and such that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(0,1) = 0$.

Given a transformation in the universe

$$T\colon \mathcal{U} \to \mathcal{U}$$

T can be extended to $\mathcal{F}(\mathcal{U})$ by defining for each $A \in \mathcal{F}(\mathcal{U})$ the fuzzy set

$$T(A)(x) = A(T(x)).$$

How inclusion is usually measured?

In set theory, a set A is included in another B if every element of A is also in B.

Such relationship can be represented in first order logic as:

$$(\forall t)A(t) \rightarrow B(t)$$

Therefore, is not strange that the most usual way to measures inclusion in the literature leads to one formula of the type

$$\mathcal{I}(A,B) = \inf\{I(A(t), B(t)) \mid t \in \mathcal{U}\}$$

where $I\colon [0,1] \times [0,1] \rightarrow [0,1]$ is an implication operator.

The set of indexes of inclusion

Instead of considering implication operators as parameters in the measurement of inclusions, we propose to consider them like indexes.

Then, we have the following set of indexes.

$$\overline{\Omega} = \{f : [0,1] \to [0,1] \mid f \text{ is increasing and } f(x) \le x\}$$

and define

Definition

Given two fuzzy sets A and B, we say that *A is f-included in B* (denoted by $A \subseteq_f B$) for $f \in \overline{\Omega}$ if and only if the inequality

$$f(A(u)) \le B(u)$$

holds for all $u \in \mathcal{U}$.

The set of indexes of inclusion

The relationship with the standard measures of inclusion

The f-inclusion is related to the formula

$$\mathcal{I}(A,B) = \inf\{I(A(t),B(t)) \mid t \in \mathcal{U}\}$$

when I is a residuated implications.

For a residuated implication I there exists a t-norm T such that:

$$I(a,b) \geq c \quad \iff \quad b \geq T(c,a)$$

for all $a,b,c \in [0,1]$. Thus

$$\bigwedge_{u\in\mathcal{U}} I(A(u),B(u)) \geq \alpha \iff I(A(u),B(u)) \geq \alpha \text{ for all } u \in \mathcal{U}$$
$$\iff B(u) \geq T(A(u),\alpha) \text{ for all } u \in \mathcal{U}$$

The set of indexes of inclusion

The relationship with the standard measures of inclusion II

$$\bigwedge_{u \in \mathcal{U}} I(A(u), B(u)) \geq \alpha \iff I(A(u), B(u)) \geq \alpha \text{ for all } u \in \mathcal{U}$$

$$\iff B(u) \geq T(A(u), \alpha) \text{ for all } u \in \mathcal{U}$$

The last inequality is in accordance with the notion of f-inclusion since the function $f_\alpha \colon [0,1] \to [0,1]$ defined by

$$f_\alpha(x) = T(x, \alpha)$$

is monotonic and $f_\alpha(x) \leq x$ for all $x \in [0,1]$.

So we are able to represent the restriction imposed by the equality

$$\inf\{I(A(t), B(t)) \mid t \in \mathcal{U}\} = \alpha$$

by means of the notion of f-inclusion.

The set of indexes of inclusion

Are they really indexes?

Proposition

Let A be a fuzzy set, then $A \subseteq_f A$ for all $f \in \overline{\Omega}$.

Proposition

Let A, B and C be three fuzzy sets and let $f, g \in \overline{\Omega}$. Then, $A \subseteq_f B$ and $B \subseteq_g C$ implies $A \subseteq_{g \circ f} C$.

Proposition

Let A and B be two fuzzy sets such that $A \subseteq_f B$ and $B \subseteq_g A$. Then:

$$|A(u) - B(u)| \leq \sup_{x \in [0,1]} \left\{x - f(x), x - g(x)\right\}$$

for all $u \in \mathcal{U}$.

The set of indexes of inclusion

Are they really indexes?

Proposition

Let A, B, C and D be fuzzy sets such that $A(u) \leq B(u)$ and $C(u) \leq D(u)$ for all $u \in \mathcal{U}$. Then $B \subseteq_f C$ implies $A \subseteq_f D$

Proposition

Let A and B be two fuzzy sets and let $f, g \in \overline{\Omega}$ such that $f \geq g$. Then $A \subseteq_f B$ implies $A \subseteq_g B$.

Proposition

Every pair of fuzzy sets A and B satisfies the relation $A \subseteq_0 B$.

The set of indexes of inclusion

Are they really indexes?

Proposition

Let A and B be two fuzzy sets. The following statements are equivalent:

1. $A \subseteq_{id} B$.
2. $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.
3. $A \subseteq_f B$ for all $f \in \overline{\Omega}$.

Proposition

Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. Then there exist $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$ if and only if the only f-inclusion of A in B is $\subseteq_0$.

The index of inclusion
Definition

Given two fuzzy sets A and B, the following set

$$\{f \in \overline{\Omega} \mid A \subseteq_f B\}$$

is closed under supremum.

Therefore, **its greatest element (denoted hereafter by f_{AB}) seems to be the most appropriated f-index of inclusion for the relation $A \subseteq B$.**

Moreover, such a mapping is determined by the following theorem.

Theorem

Let A and B be two fuzzy sets. Then, the greatest element of $\{f \in \Omega \mid A \subseteq_f B\}$ is

$$f_{AB}(x) = \min\{x, \inf_{u \in \mathcal{U}}\{B(u) \mid x \leq A(u)\}\}$$

Measures of inclusion

Sinha-Dougherty Axioms

D. Sinha and E. R. Dougherty.
Fuzzification of set inclusion: Theory and applications.
Fuzzy Sets and Systems, 55(1):15–42, 1993.

A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is an SD-inclusion measure if it satisfies the following 8 axioms

[(SD1)] $\mathcal{I}(A,B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

[(SD2)] $\mathcal{I}(A,B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.

[(SD3)] If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$

Measures of inclusion

Sinha-Dougherty Axioms

[(SD4)] If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $\mathcal{I}(C, A) \leq \mathcal{I}(B, A)$.

[(SD5)] If $T : \mathcal{U} \to \mathcal{U}$ is a bijective transformation on the universe, then

$$\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B)).$$

[(SD6)] $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.

[(SD7)] $\mathcal{I}(A \cup B, C) = \min\{\mathcal{I}(A, C), \mathcal{I}(B, C)\}$.

[(SD8)] $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$

Measures of inclusion
Kitainik Axioms

L. M. Kitainik.
Fuzzy Inclusions and Fuzzy Dichotomous Decision Procedures,
Springer Netherlands, Dordrecht, pages 154–170.1987.

A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is a K-inclusion measure if it satisfies the following 5 axioms

[(K1)] $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.

[(K2)] $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$.

Measures of inclusion

Kitainik Axioms

[(K3)] If $T : \mathcal{U} \to \mathcal{U}$ is a bijective transformation on the universe, then

$$\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B)).$$

[(K4)] If A and B are crisp then

$$\mathcal{I}(A, B) = 1 \text{ if and only if } A \subseteq B.$$

[(K5)] If A and B are crisp then

$$\mathcal{I}(A, B) = 0 \text{ if and only if } A \not\subseteq B.$$

Measures of inclusion

Young Axioms

V. R. Young.
Fuzzy subsethood.
Fuzzy Sets and Systems, 77(3):371–384, 1996.

A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is called *Y-inclusion relation* if it satisfies the following 4 axioms:

[(Y1)] $\mathcal{I}(A,B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

[(Y2)] If $A(u) \geq 0.5$ for all $u \in \mathcal{U}$, then $\mathcal{I}(A, A^c) = 0$ if and only if $A = \mathcal{U}$; i.e., $A(u) = 1$ for all $u \in \mathcal{U}$.

[(Y3)] If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $\mathcal{I}(C, A) \leq \mathcal{I}(B, A)$ for all fuzzy set $A \in \mathcal{F}(\mathcal{U})$.

Measures of inclusion

Young Axioms and Fan-Xie-Pei version

[(Y4)] If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $\mathcal{I}(A, B) \leq \mathcal{I}(A, C)$ for all fuzzy set $A \in \mathcal{F}(\mathcal{U})$.

J. Fan, W. Xie, and J. Pei.
Subsethood measure: New definitions.
Fuzzy Sets and Systems, 106(2):201–209, 1999.

proposes to change the fourth axioms in the Young's definition by

[(FX4)] If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $\mathcal{I}(A, B) \leq \mathcal{I}(A, C)$.

Measures of inclusion

Young Axioms and Fan-Xie-Pei version

[(Y4)] If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then,
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Measures of inclusion

Strong Fan-Xie-Pei Axioms

J. Fan, W. Xie, and J. Pei.
Subsethood measure: New definitions.
Fuzzy Sets and Systems, 106(2):201–209, 1999.

A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is said to be a *strong FX-inclusion relation* if it satisfies the following axioms

[(sFX1)] $\mathcal{I}(A,B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

[(sFX2)] If $A \neq \varnothing$ and $A \cap B = \varnothing$ then, $\mathcal{I}(A,B) = 0$.

[(sFX3)] If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then,
$\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ and $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$

Measures of inclusion

Weak Fan-Xie-Pei Axioms

J. Fan, W. Xie, and J. Pei.
Subsethood measure: New definitions.
Fuzzy Sets and Systems, 106(2):201–209, 1999.

A mapping $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ is said to be a *weak FX-inclusion relation* if it satisfies the following axioms

[(wFX1)] $\mathcal{I}(\varnothing,\varnothing) = \mathcal{I}(\varnothing,\mathcal{U}) = \mathcal{I}(\mathcal{U},\mathcal{U}) = 1$; where $\mathcal{U}(u) = 1$ for all $u \in \mathcal{U}$.

[(wFX2)] $\mathcal{I}(A,\varnothing) = 0$

[(wFX3)] If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then,
$\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ and $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$.

Axioms (SD1), (Y1), (K4), (sFX1) and (wFX1)

Axioms (SD1), (Y1) and (K4) are equivalent to require that

"the degree of inclusion of A in B is 1 if and only if A is contained in B in Zadeh's sense".

The Zadeh's inclusion is given by

$$A \subseteq B \text{ if and only if } A(u) \leq B(u) \text{ for all } u \in \mathcal{U}.$$

Proposition

Let A and B be two fuzzy sets. Then, $f_{AB} = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

Note that (K4) and (wFX1) are weaker assumptions than Zadeh's inclusion, and therefore, they are also satisfied by the f-index.

Axioms (SD2), (K5), (Y2), (sFX2) and (wFX2).

Null inclusions

On the one hand the f-index of inclusion does not satisfy axioms (Y2), (sFX2) and (wFX2).

On the other hand, axiom (K5) always holds and axiom (SD2) holds when the universe considered is finite.

Proposition

Let A and B be two crisp sets then, $f_{AB} = 0$ if and only if $A \not\subseteq B$.

Proposition

Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$. $f_{AB} = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.

AXIOMS (SD3), (SD4), (Y3),(Y4), (FX4), (sFX3) AND (wFX3).

Monotonicity

All the axioms about monotonicity are satisfied by the f-index.

In fact, the axioms (SD3) and (SD4) are basically described by the following result.

Proposition

Let A, B and C be three fuzzy sets:

- *if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $f_{AB} \leq f_{AC}$;*
- *if $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $f_{CA} \leq f_{BA}$.*

The rest of axioms, namely (Y3),(Y4), (FX4), (sFX3) and (wFX3) also hold for the f-indexes since are weaker forms of (SD3) and (SD4).

Axioms (SD5) and (K3).

Transformation invariance

Let us recall that the axiom (SD5) and (K3) state that for any fuzzy inclusion $\mathcal{I}$, if $T : \mathcal{U} \to \mathcal{U}$ is a transformation (i.e. a one-to-one mapping) on the universe, then

$$\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$$

for all fuzzy sets A and B.

Proposition

Let A and B be two fuzzy sets and let $T : \mathcal{U} \to \mathcal{U}$ be a transformation on $\mathcal{U}$, then $f_{AB} = f_{T(A)T(B)}$.

Then, axioms (SD5) and (K3) are satisfied by the f-index of inclusion.

Axioms (SD6) and (K1).

The role of the complement

In general, neither the equality $f_{AB} = f_{B^c A^c}$ nor the relation $A \subseteq_f B$ implies $B^c \subseteq_f A^c$ holds for $f \in \Omega$.

However, it is possible to establish some relationships between both f-indexes via adjoint pairs.

Let us assume that the complement is defined by a negation operator n, then

Proposition

Let A and B be two fuzzy sets and let (f, g) be an adjoint pair. Then $A \subseteq_f B$ if and only if $B^c \subseteq_{n \circ g \circ n} A^c$.

Theorem

Let A and B be two fuzzy sets on a finite universe $\mathcal{U}$. Then, $(f_{AB}, n \circ f_{B^c A^c} \circ n)$ forms an adjoint pair.

Axioms (SD7), (SD8) and (K8).

Relationship with union and intersection

Axioms (SD7), (SD8) and (K8) require that for any fuzzy inclusion $\mathcal{I}$ and three fuzzy sets A, B and C we have the following equalities:

- $\mathcal{I}(A \cup B, C) = \min\{\mathcal{I}(A, C), \mathcal{I}(B, C)\}$
- $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$

Theorem

Let A, B and C be three fuzzy sets then,

$$f_{A \cup B, C} = \min\{f_{AC}, f_{BC}\} \quad \textit{and} \quad f_{A, B \cap C} = \min\{f_{AB}, f_{AC}\}.$$

Conclusions

- We have shown that for a finite universe all the axioms of Sinha-Dougherty (and therefore also those of Kitainik) hold except the one related to the complement (SD6).
- With respect to the complements, we have shown a natural relationship between the f-index of A in B and the one of B^c in A^c by means of Galois connections.

As future work it would be interesting

- to continue the motivation of the f-index of inclusion as a convenient representation of the relationship $A \subseteq B$;
- to define an f-index of similarity from the f index of inclusion;
- and to establish relationships with the n-weak contradiction:

H. Bustince, N. Madrid, and M. Ojeda-Aciego. The notion of weak-contradiction: definition and measures. *IEEE Transactions on Fuzzy Systems*, 23(4):1057–1069, 2015.

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